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A MATHEMATICAL PROOF OF S. SHELAH'S THEOREM ON THE MEASURE PROBLEM AND RELATED RESULTS

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ABSTRACT
Recently, S. Shelah proved that an inaccessible cardinal is necessary to build a model of set theory in which every set of reals is Lebesgue measurable. We give a simpler and metamathematically free proof of Shelah's result. As a corollary, we get an elementary proof of the following result (without choice axiom): assume there exists an uncountable well ordered set of reals, then there exists a non-measurable set of reals. We also get results about Baire property, K_σ-regularity and Ramsey property.

§0. Introduction

Recently, S. Shelah ([5]) has given an unexpected answer to Solovay's question ([7]): is the theory ZF + DC + "every set of reals is measurable" equiconsistent with ZF?[†] Actually, Shelah proved the following result, by means of combinatorial ideas and elaborated metamathematical methods:

THEOREM 1 (S. Shelah). *Assume* ZF + DC + "*every* Σ^1_3 *set of reals is measurable*". *Then* $\aleph_1$ *is an inaccessible cardinal in the constructible universe L.*

So the theory ZF + DC + "every Σ^1_3 set of reals is measurable" is already strictly stronger than ZF.

By using S. Shelah's combinatorials ideas, but by working on 2^ω instead of $\aleph_1$, we get a simpler and metamathematically free proof of S. Shelah's theorem. Furthermore we obtain new results about other properties of sets of reals, namely the Baire property, the K_σ-regularity and the Ramsey property. Let us recall the following definitions:

[†] Throughout the paper "measurable" means "Lebesgue measurable".

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DEFINITION 1. (i) Recall that a subset of ω^ω is K_σ if it is the union of a countable sequence of compact subsets of ω^ω. A subset X of ω^ω is K_σ-regular if: either X is contained in a K_σ set, or X contains a closed set which is not K_σ.

(ii) If x is a countable set, let $[x]^\omega$ be the set of infinite subsets of x. A subset X of $[\omega]^\omega$ is Ramsey if there exists an element x of $[\omega]^\omega$ such that $[x]^\omega \subseteq X$ or $[x]^\omega \subseteq [\omega]^\omega \setminus X$.

In Solovay's model every set of reals is measurable and has the Baire property, and also every set of reals is K_σ-regular (A. Louveau [1]) and is Ramsey (A.R.D. Mathias [3]).

(By "set of reals" we mean respectively subset of 2^ω, subset of ω^ω, or subset of $[\omega]^\omega$, depending on the property we consider.)

Now let us state our result which is a generalization of Shelah's theorem:

THEOREM 2. *Assume* ZF + DC + "*every* $\boldsymbol{\Sigma}^1_2$ *set of reals is measurable*" *and either*

(i) *every* $\boldsymbol{\Sigma}^1_3$ *set of reals is measurable, or*

(ii) *every* $\boldsymbol{\Sigma}^1_3$ *set of reals has the Baire property, or*

(iii) *every* $\boldsymbol{\Sigma}^1_3$ *set of reals is* K_σ*-regular, or*

(iv) *every* $\boldsymbol{\Sigma}^1_3$ *set of reals is Ramsey.*

Then $\aleph_1$ *is an inaccessible cardinal in the constructible universe* L.

REMARK. The hypothesis "every $\boldsymbol{\Sigma}^1_2$ set is measurable" cannot be replaced by "every Σ^1_2 set is measurable" or by "every $\boldsymbol{\Sigma}^1_2$ has the Baire property" or by "every $\boldsymbol{\Sigma}^1_2$ is K_σ-regular" because of the following unpublished result of S. Shelah and J. Stern ([8]):

THEOREM 3 (S. Shelah and J. Stern). *Assume* ZF *is consistent. Then the following theory is consistent*: ZF + DC + $\aleph_1^L = \aleph_1$ + "*every ordinal definable set of reals is measurable*" + "*every set of reals has the Baire property*" + "*every set of reals is* K_σ*-regular*".

The problem to replace in Theorem 2 "every $\boldsymbol{\Sigma}^1_2$ is measurable" by "every $\boldsymbol{\Sigma}^1_2$ set is Ramsey" is still open.

As a corollary of the lightface version of Theorem 2, we get:

THEOREM 4. *Assume* ZF + DC + MA + $2^{\aleph_0} > \aleph_1$ + $\aleph_1^L = \aleph_1$. *Then there exists*:

(i) *a non-measurable* Σ^1_3 *set of reals,*

(ii) *a* Σ^1_3 *set of reals without the Baire property,*

(iii) *a non-*K_σ*-regular* Σ^1_3 *set of reals,*

(iv) *a non-Ramsey* Σ^1_3 *set of reals.*

REMARK. Parts (i) and (ii) are unpublished results of Galvin, Roirman, Shelah, Solovay, Woodin.

We get also the following:

THEOREM 5. *Assume* ZF + DC + "*there is an uncountable well ordered set of reals*". *Then there is a non-measurable set of reals.*

REMARK. S. Shelah has announced this last result suggesting a proof using a combination of his proof of Theorem 1 and J. H. Silver's proof of a theorem about coanalytic equivalence relations ([6]). We have a quite elementary proof.

We would like to close this introduction by thanking Jacques Stern, Ramez Sami and Alain Louveau for their remarks.

§1. Rapid filters

For proving Theorems 2, 4 and 5 we have to build sets of reals not having the property of measurability, the Baire property, the K_σ-regularity and the Ramsey property.

The classical constructions use the axiom of choice or at least the existence of a non-trivial ultrafilter on ω. Let us recall the following facts:

THEOREM 6. *Let* $\mathcal{U}$ *be a non-trivial ultrafilter on* ω. *Then*

(i) $\mathcal{U}$ *viewed as a subset of* 2^ω *is not measurable*;

(ii) $\mathcal{U}$ *viewed as a subset of* 2^ω *doesn't have Baire property*;

(iii) $\mathcal{U}$ *viewed as a subset of* ω^ω (*indentifying infinite subsets of* ω *with their increasing enumeration*) *is not* K_σ-*regular* (A. Louveau [1]);

(iv) *let* $f:[\omega]^\omega \to [\omega]^\omega$ *be the following continuous function*: *if* $x \in [\omega]^\omega$ *let* $(x_n)_{n\in\omega}$ *be its increasing enumeration and let* $f(x) = \bigcup_{n\in\omega} [x_{2n}, x_{2n+1}[$. *Then* $f^{-1}(\mathcal{U})$ *is not Ramsey* (A.R.D. Mathias [2]).

Of course we can't use these results. But it turns out that it is possible to replace the ultrafilter by a "rapid filter" which will play the same role. This notion is due to G. Mokobodzki [4]:

DEFINITION 2. A non-trivial filter $\mathcal{F}$ on ω is *rapid* if for every $\varphi:\omega \to \omega$ there is a set $F \in \mathcal{F}$ such that for every $k \in \omega$:

$$\overline{\overline{F \cap \varphi(k)}} \leqq k.$$

(We identify an integer with the set of its predecessors.)

THEOREM 7. *Let $\mathscr{U}$ be a rapid filter on ω. Then assertions* (i), (ii), (iii), (iv) *of Theorem* 6 *are true.*

Parts (i), (ii), (iii) are due to M. Talagrand ([9]). Part (iv) is due to A.R.D. Mathias ([2]).

Classically, constructions of rapid filters were done using AC + CH (or Martin's axiom). We show another way to construct a rapid filter. This construction is inspired by the combinatorials ideas of S. Shelah.

From now on, X is a well ordered subset of 2^ω of type $\aleph_1$, with ordering $<_X$.

DEFINITION. (i) For $\alpha, \beta \in 2^\omega, \alpha \neq \beta$ we let:

$$h(\alpha, \beta) = \operatorname{Inf}\{n \in \omega; \alpha(n) \neq \beta(n)\}.$$

(ii) Let R be some equivalence relation on 2^ω. We associate to R a set $Z_R \subset \omega$ by:

$$Z_R = \{h(\alpha, \beta); \alpha \in X, \beta \in X, \alpha \neq \beta, R(\alpha, \beta)\}.$$

(iii) We define $\mathscr{F}_X$ the filter generated by the sets Z_R, for R a Borel equivalence relation on 2^ω with countably many classes.

PROPOSITION 1. *$\mathscr{F}_X$ is a non-trivial filter on ω, and if $A \in \mathscr{F}_X$ and $n \in \omega$ then $A \cap \{k \in \omega;\ k \geqq n\} \in \mathscr{F}_X$.*

PROOF. At first let us remark that for any R_1, R_2, $Z_{R_1} \cap Z_{R_2} \supset Z_{R_1 \cap R_2}$. Let us show that each Z_R is infinite: as X has cardinality $\aleph_1$ and R countably many classes, one of the R-classes on X, say Y, is infinite. The function h maps $[Y]^2$ (the set of pairs of elements of Y) into Z_R. So, by Ramsey's theorem, if Z_R is finite then there is an infinite homogeneous set. But by definition of h, any homogeneous set is of cardinality at most 2.

Let $A \in \mathscr{F}_X$ and $n \in \omega$. Let R be such that $A \supset Z_R$. Then $A \cap \{k \in \omega;\ k \geqq n\} \supset Z_{R'}$ where R' is defined by:

$$R'(x, y) \leftrightarrow R(x, y) \wedge x \restriction n = y \restriction n.$$

DEFINITION 3. For H a subset of $2^\omega \times 2^\omega$ we let:

$$H(X) = \bigcup_{\alpha \in X} H_\alpha,$$

where H_α, the section of H at α, is defined by:

$$H_\alpha = \{\beta \in 2^\omega; (\alpha, \beta) \in H\}.$$

Now comes the key proposition.

PROPOSITION 2. *Assume the following hypothesis:*

(N) *For every G_δ subset of $2^\omega \times 2^\omega$ with null sections H, $H(X)$ is null.*[†]
Then $\mathcal{F}_X$ is a rapid filter.

PROOF. Choose recursively a family $(A(s,l,j))_{s\in 2^{<\omega}, l\in\omega, j\in\omega}$ of basic open sets in 2^ω, independent in the sense of measure, with $m(A(s,l,j)) = 2^{-(l+j)}$ (m denotes the Lebesgue measure). To each increasing function $\varphi:\omega\to\omega$, we associate a set $H^\varphi \subset 2^\omega \times 2^\omega$ by:

$$(\alpha,\beta)\in H^\varphi \leftrightarrow \forall j\ \exists j'\geqq j\ \exists l\geqq j'\ \beta\in A(\alpha\restriction\varphi(l), l, j').$$

H^φ is a G_δ set with null sections, so by (N), $H^\varphi(X)$ is null. We now use the following lemma:

LEMMA 2. *Let E be a null subset of 2^ω. There is a closed subset B of 2^ω such that $B\cap E=\emptyset$, $m(B)>0$ and for every $s\in 2^{<\omega}$, if $B\cap N_s\neq\emptyset$ then $m(B\cap N_s)\geqq 1/8^{|s|+1}$ (where $|s|$ is the length of the sequence s).*

PROOF. Let $B_0\subset 2^\omega$ be a closed set such that $B_0\cap E=\emptyset$ and $m(B_0)\geqq\frac{1}{2}$. Assume B_k is defined and let:

$$B_{k+1} = \bigcup\{B_k\cap N_s;\ s\in 2^{k+1}, m(N_s\cap B_k)\geqq 1/8^{k+1}\}.$$

Finally let:

$$B = \bigcap_{k\geq 0} B_k.$$

Let us check that B is suitable. For each $k\in\omega$ and $s\in 2^{<\omega}$, $|s|\leqq k$, one has:

$$m(N_s\cap(B_k - B_{k+1}))\leqq 2^{k+1-|s|}/8^{k+1}.$$

Hence one has:

$$m(B)\geqq\frac{1}{2}-\sum_{k\geq 0}\frac{1}{4^{k+1}}$$

and thus:

$$m(B)>0.$$

For $s\in 2^{<\omega}$, $|s|>0$ and $N_s\cap B\neq\emptyset$, one has $N_s\cap B_{|s|}\neq\emptyset$ and so $m(N_s\cap B_{|s|-1})\geqq 1/8^{|s|}$ and $N_s\cap B_{|s|} = N_s\cap B_{|s|-1}$. Hence one has:

$$m(N_s\cap B)\geqq m(N_s\cap B_{|s|}) - \sum_{k\geq|s|} m(N_s\cap(B_k - B_{k+1}))$$

$$\geqq\frac{1}{8^s}-\sum_{k\geq|s|}\frac{2^{k+1-|s|}}{8^{k+1}}$$

[†] "Null" means of measure zero.

and thus:

$$M(N_s \cap B) > 1/8^{|s|+1}.$$

Now associate with $H^\varphi(X)$ a closed set B^φ as in the lemma. The sets $\mathcal{O}_j = (\bigcup_{j' \geqq j} \bigcup_{l \geqq j'} A(\alpha \upharpoonright \varphi(l), l, j')) \cap B^\varphi$ are open in B^φ and $\bigcap_j \mathcal{O}_j = \emptyset$; so by Baire's theorem there is at least some $s \in 2^{<\omega}$ and $j \in \omega$ such that:

(1) $$B^\varphi \cap N_s \neq \emptyset \quad \text{but} \quad B^\varphi \cap N_s \cap \mathcal{O}_j = \emptyset.$$

We now define an equivalence relation R^φ. Let $\langle \quad \rangle$ be a recursive one-to-one map from $\omega \times 2^{<\omega}$ into ω such that, for every s, $\langle s \rangle \geqq |s|$. To $\alpha \in 2^\omega$ associate:

$F(\alpha)$ = the least pair $\langle j_0, s_0 \rangle$ such that (1) holds if there is any,
$F(\alpha) = \infty$ otherwise.

Then let, for α, $\beta \in 2^\omega$:

$$R^\varphi(\alpha, \beta) \leftrightarrow (F(\alpha) = \infty \text{ and } F(\beta) = \infty) \quad \text{or}$$

$$(F(\alpha) \neq \infty \text{ and } F(\alpha) = F(\beta) \text{ and } \alpha \upharpoonright \varphi(F(\alpha)) = \beta \upharpoonright \varphi(F(\beta))).$$

It is easy to check that R^φ is Borel (actually it is $\mathbf{\Sigma}^0_4$) and has countably many classes.

Let Z_φ be the corresponding set in $\mathcal{F}_X$. We claim that for every $k \in \omega$:

$$\overline{\overline{Z_\varphi \cap \varphi(k)}} \leqq k^2(3k+3)^2 2^{4k}.$$

For this we have to compute

$$Z_\varphi \cap \varphi(k) = \{h(\alpha, \beta) < \varphi(k);\ \alpha, \beta \in X, \alpha \neq \beta, R^\varphi(\alpha, \beta)\}.$$

If $\alpha, \beta \in X$ and $R^\varphi(\alpha, \beta)$ we know that there is some $j \in \omega$, $s \in 2^{<\omega}$, such that $F(\alpha) = \langle j, s \rangle$, $F(\beta) = \langle j, s \rangle$ and $\alpha \upharpoonright \varphi(\langle j, s \rangle) = \beta \upharpoonright \varphi(\langle j, s \rangle)$.

So we have, by definition of h, $h(\alpha, \beta) \geqq \varphi(\langle j, s \rangle)$, but φ is increasing, so if $h(\alpha, \beta) < \varphi(k)$ then $\langle j, s \rangle < k$. We know by definition of F and because j is less than k, that $B^\varphi \cap N_s \neq \emptyset$ and $B^\varphi \cap N_s \cap A(\alpha \upharpoonright \varphi(k), k, j) = \emptyset$ and similarly for β. Let:

$$\Delta(s, j) = \{t \in 2^{\varphi(k)};\ B \cap N_s \cap A(t, k, j) = \emptyset\},$$

$$\delta(s, j) = \overline{\overline{\Delta(s, j)}}.$$

The above remarks show that

$$\overline{\overline{Z_\varphi \cap \varphi(k)}} \leqq \sum_{\substack{\langle s, j \rangle < k \\ B^\varphi \cap N_s \neq \emptyset}} \delta(s, j)^2.$$

So we have to bound $\delta(s,j)$ (when $B^\varphi \cap N_s \neq \emptyset$). But by definition of $\Delta(s,j)$ one has:

$$B^\varphi \cap N_s \subset \bigcap_{t \in \Delta(s,j)} (2^\omega \setminus A(t,k,j))$$

which, by using the independence property, gives:

$$m(B^\varphi \cap N_s) \leqq (1 - 2^{-(k+j)})^{\delta(s,j)}.$$

But we have $B^\varphi \cap N_s \neq \emptyset$ and thus $m(B^\varphi \cap N_s) \geqq 1/8^{|s|+1}$, which, by using logarithms, gives:

$$\delta(s,j) \leqq (3|s|+3)2^{k+j}$$

and thus:

$$\overline{\overline{Z_\varphi \cap \varphi(k)}} \leqq \sum_{\langle s,j \rangle < k} ((3|s|+3)2^{k+j})^2$$

and by using that if $\langle j,s \rangle < k$, then $|s| < k$:

$$\overline{\overline{Z_\varphi \cap \varphi(k)}} \leqq k(3k+3)^2 2^{4k}.$$

Now it is quite easy to prove that $\mathcal{F}_X$ is rapid. For $k \in \omega$ let $\psi(k) = k(3k+3)^2 2^{4k}$. Given any $\varphi : \omega \to \omega$, which we may assume increasing, define φ' by:

$$\varphi'(k) = \varphi(\psi(k+1))$$

and associate $Z_{\varphi'}$. For every $k \in \omega$ we have:

$$\overline{\overline{Z_{\varphi'} \cap \varphi(\psi(k+1))}} \leqq \psi(k)$$

and it follows that, if $p \geqq \psi(0)$:

$$\overline{\overline{Z_{\varphi'} \cap \varphi(p)}} \leqq p.$$

But by Proposition 1 one can find $A \in \mathcal{F}_X$, $A \subset Z_{\varphi'}$, such that $A \cap \varphi(\psi(0)) = \emptyset$, and so for every $p \in \omega$:

$$\overline{\overline{A \cap \varphi(p)}} \leqq p.$$

Corollary (DC($\aleph_1$)). *Assume every union of $\aleph_1$ null sets is null. Then there is a rapid filter.*

Proof. Pick some $X \subset 2^\omega$ of cardinality $\aleph_1$. By Proposition 2, $\mathcal{F}_X$ is rapid.

§2. Proof of Theorems 2, 4 and 5

PROOF OF THEOREM 5. Let X be an uncountable well ordered subset of 2^ω. We may assume that X is of type $\aleph_1$. Let $<_X$ be its order. If H is a G_δ subset of $2^\omega \times 2^\omega$ with null sections, and $\alpha \in H(X)$, let $\lambda(\alpha)$ be the least $\gamma \in X$ (for $<_X$) for which $\alpha \in H_\gamma$. Then define:

$$\tilde{H}(X) = \{(\alpha, \beta) \in H(X) \times H(X); \lambda(\alpha) <_X \lambda(\beta)\}.$$

If some $\tilde{H}(X)$ is not measurable then the conclusion of Theorem 5 does hold. If all $\tilde{H}(X)$, for H a G_δ with null sections, are measurable, then (by Fubini's theorem) all $H(X)$ are null so that (N) holds. But then, by Proposition 2, $\mathcal{F}_X$ is a rapid filter, hence is not measurable (Theorem 7). In any case there is a non-measurable set.

PROOF OF THEOREM 2. We want to show that any of the assumptions implies that, for all $\alpha \in \omega^\omega$, $\aleph_1^{L[\alpha]} < \aleph_1$. Suppose not, so that $\aleph_1 = \aleph_1^{L[\alpha_0]}$ for some $\alpha_0 \in \omega^\omega$. Let $X = 2^\omega \cap L[\alpha_0]$. X then admits a good Σ_2^1 well ordering (that means, the set $\{(x, y); y \in X$ and x codes the set of predecessors of $y\}$ is Σ_2^1), and it is not hard to see that the sets $\tilde{H}(X)$ for H a G_δ are Σ_2^1 hence measurable by hypothesis. As above, it follows that (N) holds. But then, by Proposition 2, the corresponding $\mathcal{F}_X$ is a rapid filter.

Moreover, one checks that $\mathcal{F}_X$ is $\Sigma_3^1(\alpha_0)$, so that Theorem 7 gives the desired contradiction.

The preceding proof also gives Theorem 4 by noting that $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$ implies that all Σ_2^1 sets are measurable. It also gives that results hold at higher levels of projective hierarchy. For example, if Σ_n^1 sets are measurable ($n \geqq 2$) and there is a set X with a Σ_n^1 good well ordering of type $\aleph_1$, then the corresponding $\mathcal{F}_X$ is a Σ_{n+1}^1 rapid filter.

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